NPS-53-81-002

NAVAL POSTGRADUATE SCHOOL

Monterey, California

SMOOTH INTERPOLATION OF SCATTERED DATA BY

LOCAL THIN PLATE SPLINES

Richard Franke

Technical Report for Period Jan 79 - Mar 81

Prepared for: Chief of Naval Research
Arlington, VA 22217

NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral Tyler F. Dedman
Superintendent

David A. Schrady
Acting Provost

The work reported herein was supported in part by the Foundation Research Program of the Naval Postgraduate School with funds provided by the Chief of Naval Research.

This report was prepared by:

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: NPS-53-81-002	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Smooth Interpolation of Scattered Data by Local Thin Plate Splines	5. TYPE OF REPORT & PERIOD COVERED: Technical Report Jan 1979 - Mar 1981
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Richard Franke	8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Foundation Research Program, Naval Postgraduate School, Monterey, CA 93940	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Chief of Naval Research, Arlington, VA 22217	12. REPORT DATE: 26 March 1981
	13. NUMBER OF PAGES: 30
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Scattered data interpolation
Interpolation
Smooth interpolation
Bivariate interpolation
Multivariate approximation
Thin Plate Splines

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An algorithm and the corresponding computer program for solution of the scattered data interpolation problem is described. Given points (x_k, y_k, f_k), $k = 1, \ldots, N$ a locally defined function $F(x,y)$ which has the property $F(x_k, y_k) = f_k$, $k = 1, \ldots, N$ is constructed. The algorithm is based on a weighted sum of locally defined thin plate splines, and yields an interpolation function which is differentiable. The program is available from the author.

1.0 Introduction

The problem of constructing interpolation functions for sets of scattered data in two independent variables has been treated in many papers. The survey paper on approximations to multivariate functions by Schumaker [20] contains extensive references. Other survey papers are by Barnhill [3] and Sabin [19]. The present author has surveyed and tested a number of algorithms for solution of the problem [10], [11]. Conceptually the problem is quite simple: Given points (x_k, y_k, f_k), $k = 1, \ldots, N$, with distinct (x_k, y_k), construct a function $F(x,y)$ so that $F(x_k, y_k) = f_k$, $k = 1, \ldots, N$. Generally one wants to have a smooth interpolant, $F(x,y)$, in the sense that low order partial derivatives are everywhere continuous. This is complicated for large sets of data by the fact that the interpolant (in a practical sense, to be computable) must be local, so that its value at some point (x,y) depends only on (x_k, y_k, f_k) values for which (x_k, y_k) is "close" to (x,y). A general framework for a class of such methods is given in [9], and we will discuss it briefly in Section 2.

While a large number of ideas have been proposed for solution of the problem, a much smaller number of working computer programs are readily available. These include: (1) a method based on finite element functions defined over triangles, due to Akima [1], [2], a version of which is available in the IMSL* library under the name IQHSCV, (2) a program based on a similar idea, due to Lawson [14], (3) two programs based on weighted local approximations by quadratic functions, due to Franke and Nielson [12]. A program by Little [15], and another by Nielson [18], both based on finite

*International Mathematical and Statistical Libraries, 7500 Bellaire Blvd., Houston, TX 77036.

element functions defined over triangles, will probably be available by press time. All of these programs have been tested by the author [10], [11], and most perform adequately in a variety of cases; None of them seems to have a clear edge over all the others, or to be entirely satisfactory. For certain applications, each has its good points. The choice of a method for most users will be based on subjective criteria which vary from person to person and application to application. It is not surprising this is the case in two variables since it is also the case in one variable, although perhaps to a lesser extent.

The purpose of this paper is to document an alternative scheme which performed comparably well in the previously mentioned tests. In this way the computer program will be brought to the attention of potential users, who may request it from the author. The method will supplement the currently available codes in that it is based on a different approach. It is anticipated that it will find application and approval in a variety of areas.

The theoretical background for the method is discussed in Section 2, while details of the program are outlined in Section 3. Section 4 gives information concerning usage of the program. Several examples are given in Section 5, including perspective plots of some surfaces generated by the program. Included is an example of how the variation of a parameter in the method affects the surface. General guidelines for choice of this parameter are given, even though the suggested value usually leads to satisfactory results. A general discussion of the features of the method is given, along with general guidance for use of the program.

2.0 Theoretical Background

The general idea encompassing this scheme and many others are given in [9]. Consider that the points (x_k, y_k, f_k), $k = 1, \ldots, N$ are given. Briefly, local approximations to the data are constructed, and these are then blended together by using weighting functions which form a partition of unity on R^2. We pause to give the necessary notation and definitions.

A set of functions, $w_i(x,y)$, $i = 1, \ldots, m$ is said to form a partition of unity if each $w_i(x,y) \geq 0$ and $\sum_{i=1}^{m} w_i(x,y) \equiv 1$. The w_i will be called weight functions. Let the support of w_i be denoted by S_i = closure $\{(x,y): w_i(x,y) \neq 0\}$. Let $I_i = \{k: (x_k,y_k) \in \text{Interior } (S_i)\}$. Now suppose that the functions $Q_i(x,y)$, $i = 1, \ldots, m$, are defined on S_i and have the property that they interpolate the data whose (x,y) coordinates are in Interior (S_i), i.e., if $k \in I_i$, then $Q_i(x_k,y_k) = f_k$. These functions Q_i will be called local approximations. We then consider the function $F(x,y) = \sum_{i=1}^{m} w_i(x,y)Q_i(x,y)$. Its properties are summarized in the following.

<u>Proposition</u>. The function $F(x,y) = \sum_{i=1}^{m} w_i(x,y)Q_i(x,y)$ has the following properties:

(1) Interpolation; $F(x_k,y_k) = f_k$, $k = 1, \ldots, N$.

(2) Smoothness; $F(x,y)$ is at least as smooth as the w_i and Q_i, e.g., if all of the functions w_i, Q_i, $i = 1, \ldots, m$ have continuous first derivatives, so does $F(x,y)$.

(3) Local dependence on the data; Let (x,y) be fixed, and let $J_{x,y} = \{i: w_i(x,y) \neq 0\}$, then $F(x,y)$ depends only on the (x_k,y_k,f_k) points for which $k \in (\bigcup_{i \in J_{x,y}} I_i) \cup \{i:$ some Q_j, $j \in J_{x,y}$ depends on $(x_i,y_i,f_i)\}$. In particular, we have $F(x,y) = \sum_{i \in J_{x,y}} W_i(x,y)Q_i(x,y)$.

These properties are essentially observations, but form the basis for construction of appropriate weight functions which will allow easy determination of the set $J_{x,y}$. Our construction yields a set of at most four nonzero terms in the sum defining $F(x,y)$. This provides a considerably faster process during the evaluation of the interpolant than was possible in the choices previously considered [9].

It has been implied, but is not necessary for the proposition to hold, that many of the weight functions, w_i, have finite support. In order for the local approximations $Q_i(x,y)$ to actually be local, this will likely be the case. Therefore we think of weight functions whose support, S_i, contains relatively few (x_k,y_k) points, and whose support is often local. In order for the interpolant to be defined on all of R^2, some sets S_i must not be finite, and typically would contain points (x_k,y_k) on or near the boundary of the convex hull of the set of points $\{(x_k,y_k)\}$. With this framework and ideas in mind, we are ready to discuss the specific details of the algorithm.

3.0 Details of the Algorithm

3.1 Choice of Weight Functions

While the choice of weight functions was allowed to determine the support regions in the discussion of the previous section, it is more convenient to proceed from support regions to weight functions in the actual application. The use of support regions which are rectangles have specific advantages in terms of controlling the number of support regions in which a particular point (x,y) lies, as well as simplifying determination of those regions.

Let n_x and n_y be given positive integers, and let finite values of $\tilde{x}_0 < \tilde{x}_1 < \tilde{x}_2 < \ldots < \tilde{x}_{n_x} < \tilde{x}_{n_x+1}$ and $\tilde{y}_0 < \tilde{y}_1 < \tilde{y}_2 < \ldots < \tilde{y}_{n_y} < \tilde{y}_{n_y+1}$ be given. For each $i = 1, 2, \ldots, n_x$ and $j = 1, 2, \ldots, n_y$ let $r_{i,j} = [\tilde{x}_{i-1}, \tilde{x}_{i+1}] \times [\tilde{y}_{j-1}, \tilde{y}_{j+1}]$.

Let $H_3(s) = 1 - 3s^2 + 2s^3$, the Hermite cubic satisfying $H_3(0) = 1$, $H_3(1) = H_3'(0) = H_3'(1) = 0$. We then define piecewise cubics with continuous first derivatives, which are nonzero only on two adjacent intervals, and satisfy

$$v_i(x_j) = \delta_{ij}\ , \quad i,j = 1, 2, \ldots, n_x$$

$$u_j(y_i) = \delta_{ji} \quad i,j = 1, 2, \ldots, n_y.$$

In particular we take

$$v_1(x) = \begin{cases} 1 & , \quad x < \tilde{x}_1 \\ H_3\left(\dfrac{x - \tilde{x}_1}{\tilde{x}_2 - \tilde{x}_1}\right) & , \quad \tilde{x}_1 \le x < \tilde{x}_2 \\ 0 & , \quad x \ge x_2 \end{cases}$$

$$v_i(x) = \begin{cases} 0 & , \quad x < \tilde{x}_{i-1} \\ 1 - v_{i-1}(x) & , \quad \tilde{x}_{i-1} \le x < \tilde{x}_i \\ H_3\left(\dfrac{x - \tilde{x}_i}{\tilde{x}_{i+1} - \tilde{x}_i}\right) & , \quad \tilde{x}_i \le x < \tilde{x}_{i+1} \\ 0 & , \quad x \ge \tilde{x}_{i+1} \end{cases}$$

for $i = 2, \ldots, n_x - 1$

$$v_{n_x}(x) = \begin{cases} 0 & , \quad x < \tilde{x}_{n_x - 1} \\ 1 - v_{n_x - 1}(x) & , \quad \tilde{x}_{n_x - 1} \le x < \tilde{x}_{n_x} \\ 1 & , \quad x \ge \tilde{x}_{n_x} \end{cases}$$

The $u_i(y)$ are dual. Then if we define

$$W_{i,j}(x,y) = v_i(x)u_i(y) \quad , \quad i = 1, \ldots, n_x\ , \quad j = 1, \ldots, n_y,$$

it is easily observed that $W_{i,j}$ has support $r_{i,j}$, except for when $i = 1$ or n_x, or $j = 1$ or n_y, when the support extends to ∞ in one or both variables. We denote the support of $W_{i,j}$ by $R_{i,j}$. Further, we note that the $W_{i,j}$ form a partition of unity on R^2.

Since the $\tilde{x}_i$ and $\tilde{y}_j$ values give rise to a grid on the plane, we call them grid values. The choice of grid values $\tilde{x}_i$ and $\tilde{y}_j$ depend on the data. They may be specified by the user, but the default option is for them to be determined by the program. In the default mode the user specifies a parameter, NPPR (for number of points per region), the suggested value being about 10. The program will then determine the grid values so that the anticipated number of (x_k,y_k) points in each rectangle $R_{i,j}$ will be approximately NPPR. For data which is not somewhat uniformly distributed the actual numbers may vary considerably. However, for most situations we have encountered, the process is quite adequate.

An equal number of grid values is taken in each direction, i.e., $n_x = n_y$. Because we want NPPR points per rectangle, each subrectangle $(x_i,x_{i+1}) \times (y_j,y_{j+1})$ should have $\frac{1}{4}$ NPPR points. Since $n_x = n_y$, we want $(n_x + 1)^2 \frac{1}{4}$ NPPR = N, leading us to take n_x to be the nearest integer to $(4N/\text{NPPR})^{1/2} - 1$.

The grid values $\tilde{x}_i$, are now chosen so that there are approximately equal numbers of points from the values x_k, $k = 1, \ldots, N$ in each interval $(\tilde{x}_i, \tilde{x}_{i+1})$. Specifically, let $\hat{x}_k$ denote the x_k values arranged in non-decreasing order. Consider the points $(0,\hat{x}_1)$, $(1,\hat{x}_2)$, ..., $(N-1, \hat{x}_N)$, and let $g(t)$ be the piecewise linear interpolant for them. Divide the interval $(0,N-1)$ into n_x+1 subintervals of length $\Delta = \frac{N-1}{n_x+1}$. The values of $\tilde{x}_i$ are determined by taking them to be the values of g at the endpoints of the subintervals, i.e., $\tilde{x}_i = g(i\ )$, $i = 0, 1, 2, \ldots, n_x+1$. The $\tilde{y}_j$ are determined in dual fashion. This process results in the grid values and hence the rectangles, being symmetric if the (x_k,y_k) points are symmetric. In addition, the relative positions of grid values are unchanged by linear displacements and stretching in each variable.

When chosen in the above fashion, the location of the lines is not a local process in the sense that insertion of an additional point will change the boundaries of all of the rectangles. While one could argue that then the scheme is not local, we take the view that the idea of local determination of the interpolant is most important in the evaluation phase. The determination of parameters in the scheme (here, the $\tilde{x}_i$ and $\tilde{y}_j$) may be a global process. Of course, if the user specifies the grid values, he will likely be using a global process to choose them.

3.2 Choice of Local Approximations

The only constraint on the local approximations is that they interpolate the appropriate points, and that they have continuous first derivatives (at least) to assure a smooth interpolant. In the previously mentioned tests conducted by the author, a number of global interpolation schemes for scattered data were considered. In principle, any of these might be used. The choice here was made for two reasons, (1) the method scored very well in the tests, and (2) the method has an elegant and well developed mathematical theory which also has direct application to some engineering problems.

The local approximations used in this algorithm are the thin plate splines first mentioned by Harder and Desmairis [13], with theoretical developments by Duchon [4-7] and Meinguet [16], [17]. It was first developed as the solution to the problem of a thin plate which is forced to pass through certain points (the interpolation points) by application of point loads. For our purposes it is sufficient to know that the approximation is of the form

$$Q(x,y) = \sum_{k \varepsilon I} A_k d_k^2 \log d_k + a + bx + cy \ ,$$

where $I = \{k\text{:} \ \ Q \text{ is to take on the value } f_k \text{ at } (x_k, y_k)\}$, and $d_k^2 = (x - x_k)^2 + (y - y_k)^2$. The coefficients A_k, and a, b, and c are determined by the linear system of equations

$$\sum_{k\varepsilon I} A_k d_k^2 \log d_k + a + bx + cy \Big|_{(x,y) = (x_i,y_i)} = f_i \ , \quad i\varepsilon I$$

$$\sum_{k\varepsilon I} A_k = 0$$

$$\sum_{k\varepsilon I} A_k x_k = 0$$

$$\sum_{k\varepsilon I} A_k y_k = 0$$

The geometric effect of the last three equations is to suppress all terms in the approximation which grow faster than linear as distance from the interpolation points is increased. A linear system of order equal to the number of interpolation points plus three must be solved. To be nonsingular there must be at least three noncollinear points among the (x_k,y_k), $k\varepsilon I$. The system is symmetric, but not positive definite. While an equation solver designed for such systems could be used, we have found a general purpose solver, the DECOMP/SOLVE subroutines of Forsythe, Malcolm, and Moler [8] has given more reliable results.

It is easily observed that the local interpolant has continuous derivatives of all orders except at the data points, (x_k,y_k), where a logarithmic singularity occurs in the second derivatives. The interpolant has linear precision, that is, if the (x_k,y_k,f_k) points all lie on a linear function, the interpolant will reproduce it.

While the thin plate spline is invariant with respect to scaling, translation, and rotation (not all of this is obvious), the condition number of the coefficient matrix for the system of equations is dependent on scaling. To minimize difficulties with that, and to remove the effect of scaling the variables by different amounts, we transform each rectangle $r_{i,j}$ onto the unit square $[0,1]^2$, before the local approximation $Q_{i,j}$ is computed.

It remains to specify the process for determining the points (x_k, y_k, f_k) to be interpolated by the thin plate spline local approximation. Experience has shown that it is advantageous to include more points than is necessary, i.e., (x_k, y_k) which are outside of $R_{i,j}$. This tends to yield a better transition between local approximations than when only necessary points are included. The set of (x_k, y_k) points transformed into the rectangle $R = [-.1125, 1.1125]^2$ by the transformation taking $r_{i,j}$ onto $[0,1]^2$ are included. Let

$$I_{i,j} = \{k: \left(\frac{x_k - \tilde{x}_{i-1}}{\tilde{x}_{i+1} - \tilde{x}_{i-1}} , \frac{y_k - \tilde{y}_{j-1}}{\tilde{y}_{j+1} - \tilde{y}_{j-1}} \right) \varepsilon\ R\}.$$

This gives the basic set of interpolation points for the local approximation $Q_{i,j}$ associated with the rectangle $R_{i,j}$. Under certain conditions there may be fewer than the necessary three indices in $I_{i,j}$. When this happens, the set $I_{i,j}$ is augmented by including as interpolation points the necesary number of closest points to the rectangle $R_{i,j}$ (in the ℓ_∞ norm), after the points (x_k, y_k) have undergone the transformation to the unit square. The minimum number of points per rectangle is a variable, MINPTS. This has been set to 3, but may be increased if it seems desirable.

After the interpolation points for each local approximation have been determined, the local thin plate splines, $Q_{i,j}$, can be determined by calculating the coefficients. This yields

$$Q_{i,j}(x,y) = \sum_{k \varepsilon I_{i,j}} A_{i,j,k}\, d_k'^2 \log d_k' + a_{i,j} + b_{i,j}\, x' + c_{i,j}\, y'$$

where the primes denote coordinates and distance after the transformation of $r_{i,j}$ to the unit square

3.3 Properties of the Interpolant

The overall interpolant is of the form

$$F(x,y) = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} W_{i,j}(x,y) Q_{i,j}(x,y),$$

and as noted previously, there are at most four nonzero terms in the sum. Which terms are nonzero is easily determined. If $x \geq \tilde{x}_{n_x}$, set $i' = n_x$; Otherwise let i' be the smallest index so that $\tilde{x}_{i'+1} > x$. Determine j' in dual fashion from y and the $\tilde{y}_j$'s. Then, the four nonzero terms have indices (i',j'), $(i'+1,j')$, $(i',j'+1)$, and $(i'+1,j'+1)$. If $i' = 0$ or $i' = n_x$, the terms involving i' or $i'+1$, respectively, do not appear, and similarly if $j' = 0$ or $j' = n_y$, the terms involving j' or $j'+1$, respectively, do not appear.

In addition to the properties outlined in Section 2, certain other properties hold. The approximation is invariant under translation and stretching (independently in each variable). It has symmetry with respect to planes parallel to coordinate planes whenever the data has that symmetry. The approximation is not invariant under rotations, however, since the rectangles depend on the individual coordinates of the data points.

The approximation has continuous first derivatives, and jump discontinuities in the second derivatives across grid lines, as well as logarithmic singularities in the second derivatives at the data points. Plots of the surfaces generally appear to be quite smooth, however. Since the local approximations have linear precision, the overall approximation also has linear precision, i.e., if the data lies on a linear function, the interpolant is a linear function.

4.0 The Computer Program

The overall hierarchy of the subroutines is given in Diagram 1, which shows the communication links between them. No COMMON is used as array sizes are problem dependent and thus specified by the user. We briefly discuss them, mentioning important parameters.

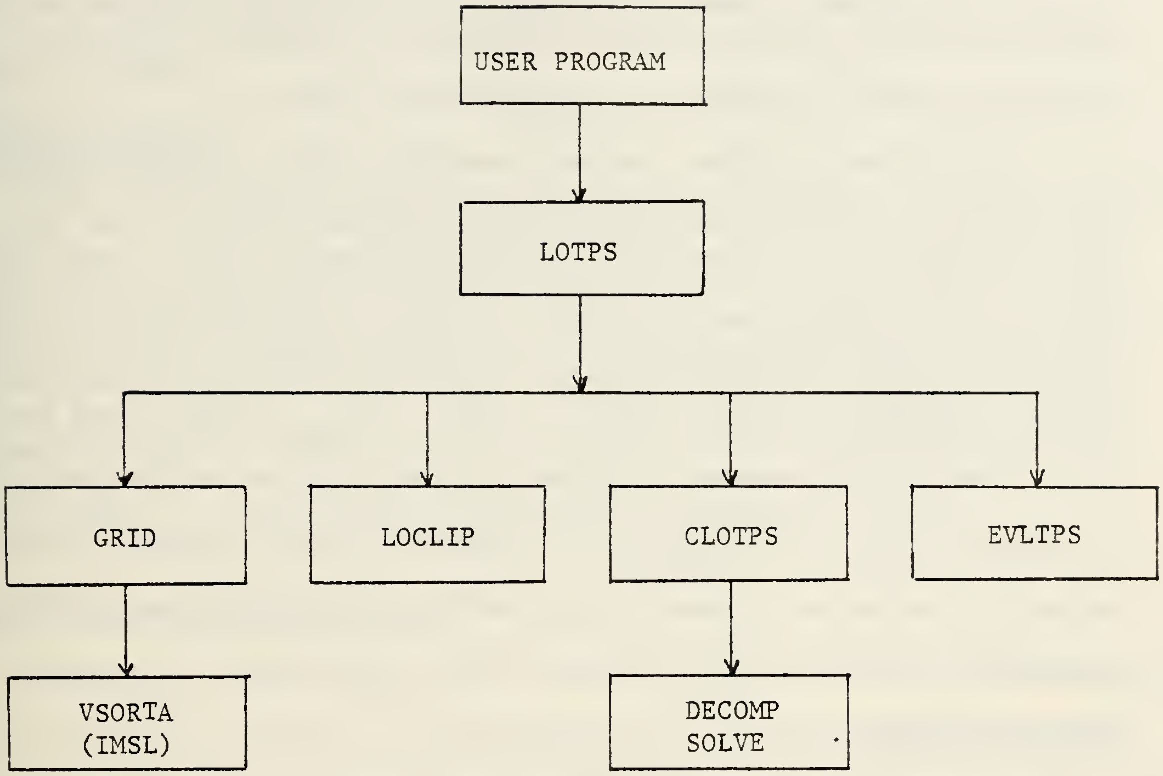

Diagram 1.

Calling Program

This program is supplied by the user and must supply the (x_k, y_k, f_k) points, plus two arrays of points $x0_i$, and $y0_j$ for the grid of points $(x0_i, y0_j)$ at which the interpolant is to be evaluated. In addition, the user must supply two workspace arrays, IWK and WK in which information calculated during preprocessing (e.g., $I_{i,j}$, $\tilde{x}_i$, $\tilde{y}_j$, and coefficients for the local approximations), is stored, and an array FO for the returned interpolant values.

The amount of storage required for the arrays IWK and WK is not known a priori. The estimated space required is about 6N for IWK and about 7N for WK. Table 1 gives exact results for several different sized problems based on random (x,y) points. Oddly distributed point sets may result in somewhat more storage being required. In any case, the precise number of locations required is returned to the calling program from Subroutine LOTPS, the only routine referenced by the user. If an insufficient number are allowed, an error return occurs.

Under the usual option, the user specifies NPPR > 0, the suggested value being 10. If the user wishes to specify the grid lines, he may do so by setting NPPR = 0, and then giving grid line information in the arrays IWK and WK, as explained in the argument description for Subroutine LOTPS. Typically one should take $\tilde{x}_0 = \min_k x_k$, $\tilde{x}_{n_x+1} = \max_k x_k$, and the dual in y. This is not necessary, although all points to be interpolated should lie in $[\tilde{x}_0, \tilde{x}_{n_x+1}]$ $[\tilde{y}_0, \tilde{y}_{n_y+1}]$. To prevent different scaling (internally) in the two variables, a square grid covering the (x_k, y_k) points could be specified.

<u>Subroutine LOTPS</u>

This subroutine provides the interface between the user's program and the set of routines implementing the method. Generally, LOTPS sets up storage areas in the arrays IWK and WK, determines parameters required by other subroutines, and calls other subroutines to (1) generate the grid, if necessary, (2) determine the interpolation points for the local approximations, (3) compute coefficients for the local approximations, (4) evaluate the interpolant at a grid of points.

<u>Subroutine GRID</u>

This subroutine selects values of $\tilde{x}_i$ and $\tilde{y}_j$ in accordance with the discussion in Section 3.

<u>Subroutine LOCLIP</u>

This subroutine determines the interpolation points for each local interpolant, in accordance with Section 3.

<u>Subroutine CLOTPS</u>

This subroutine generates the system of equations for the coefficients of the local approximations and calls an equation solver to obtain them.

Internally the routine has a maximum of 30 local interpolation points. This can easily be altered by changing two statements, as noted in the program listing.

Subroutine EVLTPS

This subroutine evaluates the interpolant on the grid of points specified by the user. Use of a grid, when that is required, facilitates the process of locating the rectangle in which a particular evaluation point is located. The $x0_i$ and $y0_i$ values should be in increasing order for maximum efficiency. Evaluation time for a grid of points should be nearly independent of N for large N, which is borne out by the timing information in Table 2.

5.0 Examples and Observations

Example 1. This example shows a typical local approximation function and is for the data in Table 1. This function is a "cardinal" function for the first point, and as such shows the effect of a nonzero value at a single point on the interpolant. The plotted surface, shown in Figure 1, is over $[0,1]^2$.

Example 2. This example is given to show a surface generated from 100 randomly generated points with the function value being obtained from an explicitly given function. The surface was generated using NPPR = 10, has rms error of about .3%, and is virtually indistinguishable from a plot of the parent surface. It is shown in Figure 2.

Examples 3, 4, 5. These examples use the same set of 60 points lying in the square $(-\frac{1}{18}, \frac{19}{18})^2$, and chosen by a pseudorandom number generator. The function is explicitly given by $f(x,y) = .1 + \frac{\sin 3(x+y)}{12(x+y)}$, and is shown in Figure 3. Figures 4, 5, and 6 show the interpolant over the square $[0,1]^2$ for NPPR values of 6, 10, and 15.

From these examples we see that NPPR = 10 works well, not too much difference is observed when NPPR is increased, but NPPR = 6 gives a less smooth appearing surface. The smaller NPPR is, the more localized the surface becomes (although NPPR = 4 will probably be the least value practicable, and some local interpolants will likely become planes due to the minimum of three interpolation points being reached). In line with this comment, very smooth surfaces with small gradients will probably be amenable to larger NPPR, while surfaces with large gradients may be best approximated by taking NPPR smaller, thus localizing the behavior.

Table 2 gives the results of a series of problems with various numbers of points and values of NPPR. The data points were chosen by a pseudorandom number generator in the square $[-\frac{1}{18}, \frac{19}{18}]^2$, and the approximation was evaluated on a 33x33 grid (of 1089 points total) on $[0,1]^2$. We observe that increasing NPPR increases execution time while decreasing the amount of storage needed. Preprocessing time should be about proportional to N^2, but apparently there is a strong linear component for small N. Preprocessing time should also be about proportional to NPPR, although this is not readily apparent. Evaluation time should be nearly independent of N for large N, and proportional to NPPR, which is approximately shown.

The automatic grid selection process works well when the data is fairly uniform in (x,y) and lies nearly in a square region. If the data is very irregular, or lies in an oblong rectangular area, it will probably be useful to explore results with a user specified grid to obtain better coverage by rectangles which are not too oblong. Limited experience has been accumulated in these situations.

6.0 How to Obtain this Program

A copy of this program, written in Fortran, and including a sample driver program, can be obtained from the author. To do so, send a (short) 1/2 inch tape to the author indicating the format desired.

Acknowledgements

This work was supported, in part, by the Foundation Research Program at the Naval Postgraduate School. The final form of the program and documentation was accomplished while the author was on sabbatical leave at Drexel University. The author extends his thanks to the Center for Scientific Computation and Interactive Graphics at Drexel for use of their Prime 400 computer system, on which all examples were run.

x	y	f
0.35	0.35	0.5
-0.05	0.25	0.0
0.10	-0.05	0.0
0.50	0.05	0.0
0.00	0.90	0.0
0.30	0.70	0.0
0.60	0.50	0.0
0.90	0.00	0.0
0.40	1.05	0.0
0.85	0.80	0.0
1.05	0.20	0.0
1.10	1.10	0.0

Table 1: Data for "Cardinal" function.

NPPR	N	$n_x = n_y$	NWKU	NIWKU	Time (Preprocessing)	Time (Evaluation)
6	60	5	334	273	4.2	14.3
6	100	7	619	506	7.5	17.6
6	500	18	3596	2893	70.5	21.2
6	1000	25	7441	6140	201.5	21.3
10	60	4	284	243	5.3	18.0
10	100	5	488	427	9.9	26.4
10	500	13	3014	2649	73.3	29.8
10	1000	19	6565	5804	202.6	31.7
15	60	3	224	199	6.8	23.0
15	100	4	414	373	12.4	32.8
15	500	11	2856	2591	91.6	39.3
15	1000	15	5920	5439	258.5	40.6

Table 2: Storage/Times for Various Size Problems.

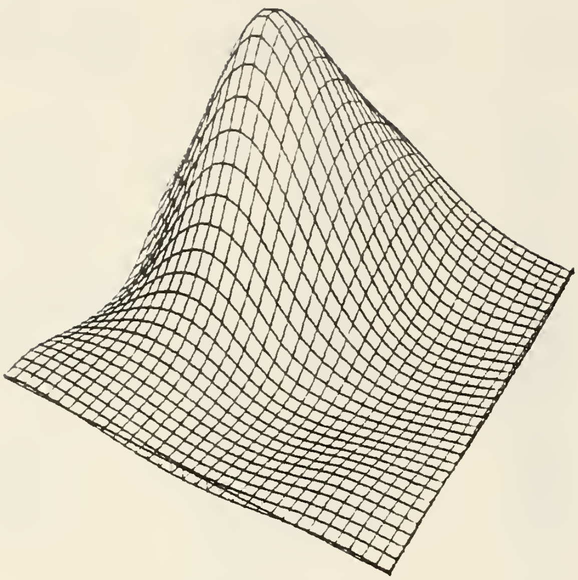

Figure 1: Cardinal Function

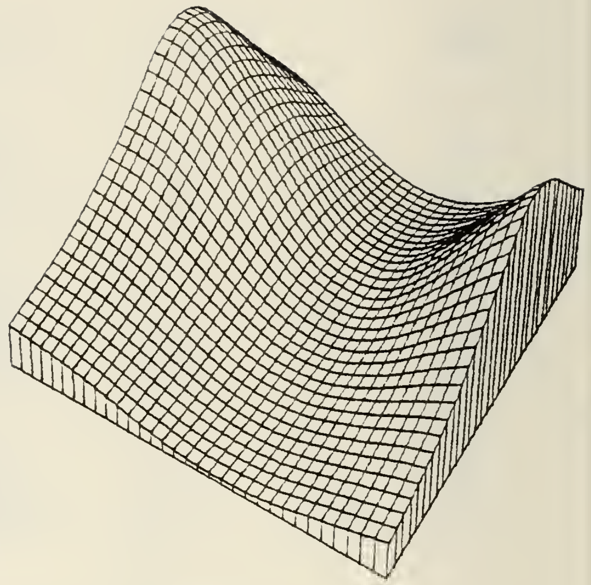

Figure 2: Saddle Function

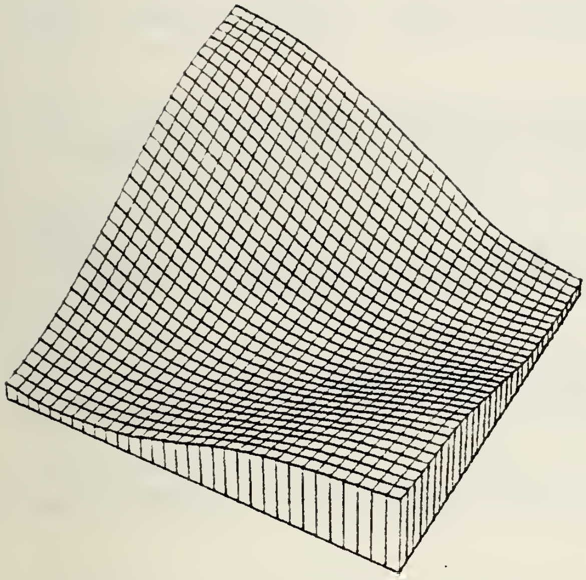

Figure 3: Parent Function

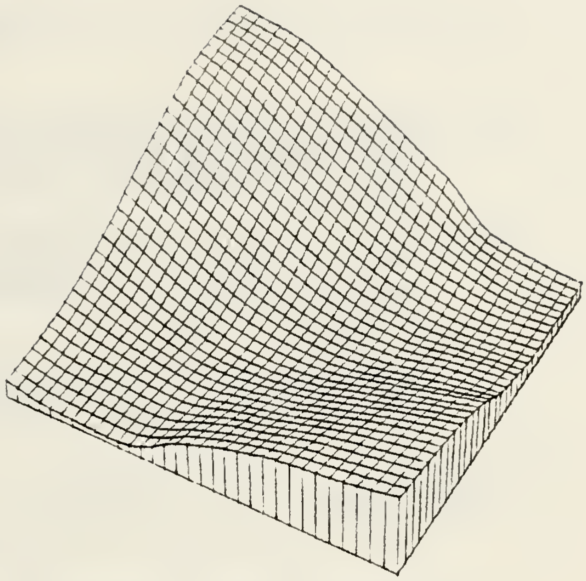

Figure 4: NPPR = 6

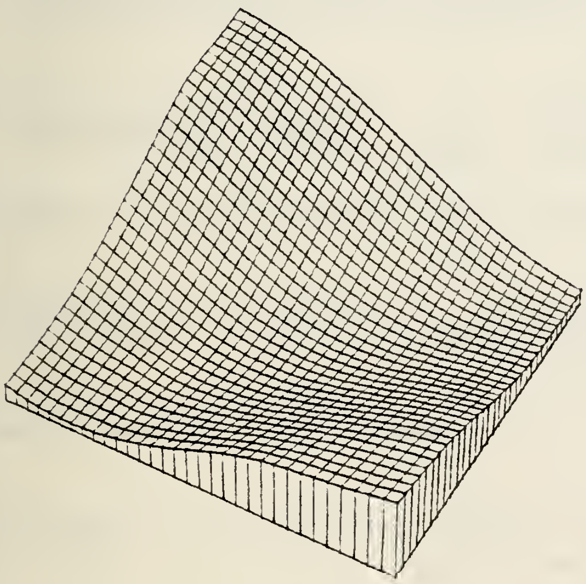

Figure 5: NPPR = 10

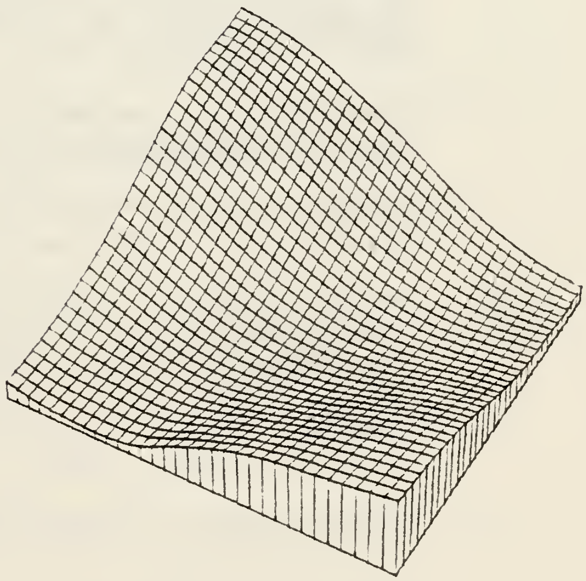

Figure 6: NPPR = 15

References

[1] AKIMA, Hiroshi - "A Method of Bivariate Interpolation and Smooth Surface Fitting for Irregularly Distributed Data Points", ACM TOMS 4 (1978) 148-159.

[2] AKIMA, Hiroshi - "Algorithm 526: Bivariate Interpolation and Smooth Surface Fitting for Irregularly Distributed Data Points", ACM TOMS 4 (1978) 160-164.

[3] BARNHILL, R.E. - "Representation and Approximation of Surfaces", pp 69-120 in Mathematical Software III, J.R. Rice, ed., Academic Press, 1977.

[4] DUCHON, Jean - "Fonctions - Spline du Type Plaque Mince en Dimencion 2", Report #231, Univ. of Grenoble, 1975.

[5] DUCHON, Jean - "Fonctions - Spline a Energie Invariate par Rotation", Report #27, Univ. of Grenoble, 1976.

[6] DUCHON, Jean - "Interpolation des fonctions de deux Variables Suivant le Principe de la Flexion des Plaques Minces", R.A.I.R.O. Analyse Numerique 10 (1976) 5-12.

[7] DUCHON, Jean - "Splines Minimizing Rotation - Invariant Semi-Norms in Sobolev Spaces", pp 85-100 in Constructive Theory of Functions of Several Variables, W. Schempp and K. Zeller, eds., Lecture Notes in Math 571, Springer, 1977.

[8] FORSYTHE, G.E.; MALCOLM, M.A.; MOLER, C.B. - Computer Methods for Mathematical Computations, Prentice-Hall, Englewood Cliffs, NJ, 1977.

[9] FRANKE, Richard - "Locally Determined Smooth Interpolation at Irregularly Spaced Points in Several Variables", JIMA 19 (1977) 471-482.

[10] FRANKE, Richard - "A Critical Comparison of Some Methods for Interpolation of Scattered Data", Naval Postgraduate School, TR#NPS-53-79-003, 1979 (Available from NTIS, #AD-A081 688/4).

[11] FRANKE, Richard - "Scattered Data Interpolation: Tests of Some Methods", to appear in Math. Comp.

[12] FRANKE, Richard; NIELSON, Gregory - "Smooth Interpolation of Large Sets of Scattered Data", Int. J. Num. Meth. Engrg. 15 (1980) 1691-1704.

[13] HARDER, R.L.; DESMARAIS, R.N. - "Interpolation Using Surface Splines", J. Aircraft 9 (1972) 189-191.

[14] LAWSON, C.L. - "Software for C^1 Surface Interpolation", pp 159-192 in Software III, J.R. Rice, ed., Academic Press, 1977.

[15] LITTLE, Frank, University of Utah CAGD Report (forthcoming).

[16] MEINGUET, Jean - "Multivariate Interpolation at Arbitrary Points Made Simple", ZAMP 30 (1979) 292-304.

[17] MEINGUET, Jean - "An Intrinsic Approach to Multivariate Spline Interpolation at Arbitrary Points", pp 163-190 in Polynomial and Spline Approximation, B.N. Sahney, ed., D. Reidel Publishing Co., 1979.

[18] NIELSON, Gregory M. - "A Method for Interpolating Scattered Data Based Upon a Minimum Norm Network", a manuscript.

[19] SABIN, M.A. - "Contouring - A Review of Methods for Scattered Data", pp 63-85 in Mathematical Methods in Computer Graphics and Design, K.W. Brodlie, ed., Academic Press, 1980.

[20] SCHUMAKER, L.L. - "Fitting Surfaces to Scattered Data", in Approximation Theory II, pp 203-268, G.G. Lorentz, C.K. Chui, L.L. Schumaker, eds., Academic Press, 1976.

```
      SUBROUTINE LOTPS (MODE,NPPR,NPI,XI,YI,FI,NXO,XO,NYO,YO,IWK,NIWK,
     1 NIWKU,WK,NWK,NWKU,FO,KER)
C
C     THIS SUBROUTINE SERVES AS A USER INTERFACE TO THE SET OF
C     SUBROUTINES THAT IMPLEMENT FRANKE'S METHOD OF SURFACE INTERPO-
C     LATION.  RECTANGLULAR REGIONS ARE USED WITH PRODUCT CUBIC
C     HERMITE WEIGHT FUNCTIONS.  THE RECTANGLES ARE CHOSEN IN AN ATTEMPT
C     TO OBTAIN ABOUT NPPR POINTS IN EACH REGION.  THE SAME NUMBER OF
C     GRID LINES IS USED IN EACH DIRECTION.  LOCAL INTERPOLATION FUNC-
C     TIONS ARE THE THIN PLATE SPLINES DESCRIBED BY DUCHON AND OTHERS.
C     A DESCRIPTION OF THE METHOD AND REFERENCES APPEAR IN
C
C     SMOOTH INTERPOLATION OF SCATTERED DATA BY LOCAL THIN PLATE SPLINES
C             NAVAL POSTGRADUATE SCHOOL REPORT  NPS-53-81-002
C                RICHARD FRANKE
C                DEPARTMENT OF MATHEMATICS
C                NAVAL POSTGRADUATE SCHOOL
C                MONTEREY, CALIFORNIA  93940
C                     (408)646-2758 / 2206
C
C        DIFFICULTIES AND SUCCESSES WITH THIS PROGRAM SHOULD BE
C        COMMUNICATED TO THE AUTHOR.
C
C     THE ARGUMENTS ARE AS FOLLOWS.
C
C        MODE - INPUT.  INDICATES THE STATUS OF THE CALCULATION.
C                 = 1,  SET UP THE PROBLEM.  COMPUTE THE COEFFICIENTS
C                       FOR THE LOCAL APPROXIMATIONS BY THIN PLATE
C                       SPLINES, AND RETURN THE GRID OF INTERPOLATED
C                       FUNCTION VALUES INDICATED BY NXO, XO, NYO, YO
C                       IN FO.
C                 = 2,  THE PROBLEM HAS BEEN SET UP PREVIOUSLY.  CALCU-
C                       LATE THE GRID OF INTERPOLATED POINTS INDICATED
C                       BY NXO, XO, NYO, YO IN FO.  THE PROGRAM ASSUMES
C                       THAT THE ARRAYS XI, YI, IWK, AND WK ARE
C                       UNCHANGED FROM THE PREVIOUS CALL.
```

```
C        NPPR - INPUT.  DESIRED AVERAGE NUMBER OF POINTS PER REGION.     LOT  37
C                       THE SUGGESTED VALUE IS TEN.  SHOULD BE AT LEAST LOT  38
C                       FOUR.  VALUES LARGER THAN FIFTEEN COULD REQUIRE LOT  39
C                       MINOR PROGRAM MODIFICATIONS TO ALLOW MORE       LOT  40
C                       STORAGE FOR THE EQUATION SOLVER.  THIS DEPENDS  LOT  41
C                       ON THE DISPOSITION OF THE POINTS.               LOT  42
C                       IF THE USER WISHES TO SPECIFY HIS OWN GRID LINESLOT  43
C                       X TILDA AND Y TILDA, HE MAY DO SO BY SETTING    LOT  44
C                       NPPR = 0 AND SETTING NECESSARY VALUES IN THE    LOT  45
C                       ARRAYS IWK AND WK, AS NOTED BELOW.  DATA WHICH  LOT  46
C                       HAS A POOR DISTRIBUTION OVER THE REGION OF      LOT  47
C                       INTEREST SHOULD PROBABLY HAVE THE GRID SPECIFIEDLOT  48
C                       THIS IS ALSO ADVISABLE IF THE X-Y POINTS OCCUR  LOT  49
C                       ON LINES.                                       LOT  50
C        NPI  - INPUT.  NUMBER OF INPUT DATA POINTS.                    LOT  51
C        XI   - \                                                       LOT  52
C        YI   - INPUT.  THE DATA POINTS (XI,YI,FI), I=1,...,NPI.        LOT  53
C        FI   - /                                                       LOT  54
C        NXO  - INPUT.  THE NUMBER OF XO VALUES AT WHICH THE INTERP-    LOT  55
C                       OLATION FUNCTION IS TO BE CALCULATED.           LOT  56
C        XO   - INPUT.  THE VALUES OF X AT WHICH THE INTERPOLATION      LOT  57
C                       FUNCTION IS TO BE CALCULATED.  THESE SHOULD     LOT  58
C                       BE IN INCREASING ORDER FOR MOST EFFICIENT       LOT  59
C                       EVALUATION.                                     LOT  60
C        NYO  - INPUT.  THE NUMBER OF YO VALUES AT WHICH THE INTERP-    LOT  61
C                       OLATION FUNCTION IS TO BE CALCULATED.           LOT  62
C        YO   - INPUT.  THE VALUES OF Y AT WHICH THE INTERPOLATION      LOT  63
C                       FUNCTION IS TO BE CALCULATED.  THESE SHOULD     LOT  64
C                       BE IN INCREASING ORDER FOR MOST EFFICIENT       LOT  65
C                       EVALUATION.                                     LOT  66
C        IWK  - INPUT AND OUTPUT.  THIS ARRAY IS OUTPUT WHEN MODE = 1   LOT  67
C                       AND IS INPUT WHEN MODE = 2.  THIS MUST BE       LOT  68
C                       AN ARRAY DIMENSIONED APPROXIMATELY 6*NPI.       LOT  69
C                       WHEN NPPR IS INPUT AS ZERO THE USER MUST        LOT  70
C                       SPECIFY THE NUMBER OF VERTICAL GRID LINES (THE  LOT  71
C                       NUMBER OF X TILDA VALUES) IN IWK(1) AND THE     LOT  72
C                       NUMBER OF HORIZONTAL GRID LINES (THE NUMBER OF  LOT  73
C                       Y TILDA VALUES) IN IWK(2).                      LOT  74
```

```
C         NIWK - INPUT.  ON ENTRY WITH MODE = 1 THIS MUST BE SET TO THE  LOT  75
C                        DIMENSION OF THE ARRAY IWK.                     LOT  76
C         NIWKU- OUTPUT.  THE ACTUAL NUMBER OF LOCATIONS USED IN THE     LOT  77
C                        ARRAY IWK.                                      LOT  78
C         WK   - INPUT AND OUTPUT.  THIS ARRAY IS OUTPUT WHEN MODE = 1   LOT  79
C                        AND IS INPUT WHEN MODE = 2.  THIS MUST BE AN    LOT  80
C                        ARRAY DIMENSIONED APPROXIMATELY 7*NPI LOCATIONS.LOT  81
C                        LOCATIONS.                                      LOT  82
C                        WHEN NPPR IS INPUT AS ZERO THE USER MUST SPECIFYLOT  83
C                        THE VALUES OF X TILDA AND Y TILDA AS FOLLOWS.   LOT  84
C                        WK(2), ... , WK(NXG+1) ARE THE NXG (= IWK(1))   LOT  85
C                        X GRID VALUES, X(I) TILDA, IN INCREASING ORDER. LOT  86
C                        TYPICALLY WK(1) = MIN X(I), ALTHOUGH IT NEED    LOT  87
C                        NOT BE.  WK(1) MUST BE LESS THAN OR EQUAL TO    LOT  88
C                        WK(2), AND SHOULD BE LESS THAN OR EQUAL TO      LOT  89
C                        MIN X(I).  WK(NXG+2) IS USUALLY MAX X(I), AL-   LOT  90
C                        THOUGH IT NEED NOT BE.  WK(NXG+2) MUST BE       LOT  91
C                        GREATER THAN WK(NXG+1), AND SHOULD BE GREATER   LOT  92
C                        THAN OR EQUAL TO MAX X(I).                      LOT  93
C                        THE VALUES OF WK(NXG+3), ... , WK(NXG+NYG+4)    LOT  94
C                        ARE THE Y GRID VALUES, Y(I) TILDA, AND MUST     LOT  95
C                        SATISFY DUAL CONDITIONS.                        LOT  96
C         NWK  - INPUT.  ON ENTRY WITH MODE = 1 THIS MUST BE SET TO THE  LOT  97
C                        DIMENSION OF THE ARRAY WK.                      LOT  98
C         NWKU - OUTPUT.  THE ACTUAL NUMBER OF LOCATIONS USED IN THE     LOT  99
C                        ARRAY WK.                                       LOT 100
C         FO   - OUTPUT.  VALUES OF THE INTERPOLATION FUNCTION AT THE    LOT 101
C                        GRID OF POINTS INDICATED BY NXO, XO, NYO, YO.   LOT 102
C                        FO IS ASSUMED TO BE DIMENSIONED (NXO,NYO) IN THELOT 103
C                        CALLING PROGRAM.                                LOT 104
```

```
C        KER   - OUTPUT.  RETURN INDICATOR.                          LOT 105
C                  = 0,  NORMAL RETURN.                              LOT 106
C                  =-1,  PROBLEM HAS NOT BEEN PREVIOUSLY SET UP (LOTPS   LOT 107
C                        CALLED WITH MODE = 1)                       LOT 108
C                  = 1,  ERROR RETURN FROM CLOTPS, SINGULAR MATRIX IN THELOT 109
C                        CALCULATION OF THE THIN PLATE SPLINES.      LOT 110
C                  = 2,  ERROR RETURN FROM CLOTPS.  SOME RECTANGLE  (I,J)LOT 111
C                        HAS MORE THAN THE ALLOWED NUMBER OF POINTS  LOT 112
C                        ASSOCIATED WITH IT.  SEE CLOTPS FOR THE FIX.    LOT 113
C                  = 3,  PREVIOUS ERROR RETURN FROM CLOTPS HAS NOT BEEN  LOT 114
C                        CORRECTED.                                  LOT 115
C                  = 4,  IWK AND/OR WK ARRAYS HAVE NOT BEEN DIMENSIONED  LOT 116
C                        LARGE ENOUGH IN THE CALLING PROGRAM.  REDIMEN-  LOT 117
C                        SION IWK AND WK TO AT LEAST THE SIZE INDICATED  LOT 118
C                        BY NIWKU AND NWKU, RESPECTIVELY.            LOT 119
C                  = 5,  MODE IS OUT OF RANGE.                       LOT 120
C                                                                    LOT 121
C     SUBPROGRAMS CALLED BY THIS ROUTINE ARE    GRID, LOCLIP, CLOTPS,    LOT 122
C     AND EVLTPS.  ALSO REQUIRED ARE  VSRTA  FROM THE IMSL LIBRARY AND   LOT 123
C     THE ROUTINES  DECOMP AND SOLVE  FROM FORSYTHE, MALCOLM, AND MOLER.LOT 124
```

DISTRIBUTION LIST

Defense Technical Information Center Cameron Station Alexandria, VA 22314	2
Dudley Knox Library Naval Postgraduate School Monterey, CA 93940	2
Dean of Research Naval Postgraduate School Monterey, CA 93940	2
Department of Mathematics	
C. O. Wilde, Chairman	1
Professor F. D. Faulkner	1
Professor A. Schoenstadt	1
Professor R. Franke	10
Naval Postgraduate School Monterey, CA 93940	
Dr. Richard Lau Office of Naval Research 1030 East Green St. Pasadena, CA 91106	1
Professor R. E. Barnhill Department of Mathematics University of Utah Salt Lake City, UT 84112	1
Professor G. M. Nielson Department of Mathematics Arizona State University Tempe, AZ 85281	1
Chief of Naval Research ATT: Mathematics Program ARlington, VA 22217	2
Rosemary E. Chang Sandia Laboratories Applied Mathematics, Division 2-8235 Livermore, CA 94550	1
L. H. Seitelman Pratt and Whitney Aircraft 400 Main Street East Hartford, CT 06108	1
William W. Bozek Engineering Data Processing Philips ECG, Inc. Johnston Street Seneca Falls, NY 13148	1

William J. Gordon 1
Department of Mathematical Sciences
Drexel University
Philadelphia, PA 19104

Loren N. Argabright 1
Department of Mathematical Sciences
Drexel University
Philadelphia, PA 19104

U196797